Advanced Mathematical Models & Applications
Vol.5, No.1, 2020, pp.38-52

HIGH-ORDER JACOBI TAU SCHEME FOR NONLINEAR VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS WITH PROPORTIONAL DELAY

H. Ansari Ouzi*

Sahand University of Technology, Tabriz, Iran

Abstract. In this paper, we develop and analyze a high-order operational Jacobi Tau scheme for the numerical solution of nonlinear Volterra integro-differential equations with proportional delay. To this end, we will present our scheme based on three simple matrices. We present the scheme by applying the shifted Jacobi polynomials as basis functions and using some simple matrix and vector operations, we show that the Tau solution of the considered equation will be obtained by solving a sparse upper triangular nonlinear algebraic system which can be solved directly by forward substitution method. Finally, some numerical examples are given to confirm the effectiveness and reliability of the proposed method.

Keywords: nonlinear Volterra integro-differential equations, proportional delay, high-order Jacobi Tau scheme, shifted Jacobi polynomials.
AMS Subject Classification: 45D05, 65R20.
Corresponding author: H. Ansari Ouzi, Sahand University of Technology, Tabriz, Iran,
e-mail: *H.Ansari.0223@gmail.com*
Received: 13 December 2019; Revised: 16 March 2020; Accepted: 31 March 2020; Published: 29 April 2020.

1 Introduction

The main concern of this paper is to introduce and analyze a well-posed approximate method based on the Jacobi Tau scheme for the numerical solution of the following nonlinear Volterra integro-differential equations with proportional delay

$$\begin{cases} \mathcal{D}y(x) = f(x) + \int_0^{qx} K(x,t)F(t,y(t))dt, & q \in (0,1),\ x \in \Omega = [0,1], \\ y^{(\zeta)}(0) = d_\zeta, \quad \zeta = 0,1,\dots,\xi-1, \end{cases} \tag{1}$$

where $f(x), K(x,t)$ and $F(t,y(t))$ are given appropriately smooth functions on their respective domains also $F(t,y(t))$ nonlinear in $y(t)$, and $y(t)$ is a solution for the main equation to be determined and d_ζ some given constants. Let $\mathcal{D}$ be a linear differential operator of order n_d with polynomial coefficients defined by

$$\mathcal{D} := \sum_{i=0}^{n_d} p_i(x)\frac{\partial^i}{\partial x^i},$$

where $p_i(x) := \sum_{j=0}^{\alpha_i} p_{ij}x^j$ and α_i is the degree of $p_i(x)$. These kinds of equations include many interesting applications for special cases such as, physics, biology, ecology, control theory and so on (Bellen & Zennaro, 2003; Iserles & Liu, 1994, 1997a,b; Ishitawa & Muroya, 2009; Muroya et al., 2003; Tohidi et al., 2013). From well-known existence and uniqueness theorems we can conclude that in (1), smooth data lead to solutions that are smooth on the entire interval. Thus spectral methods which make high-order approximate solutions for the functional equations with smooth solutions can be applied to obtain a reliable numerical solution for (1). The speed of convergence is one of the great feature of the spectral methods. Besides, the spectral methods have high rates of convergence, they also have high level of reliability.

Several analytical and numerical techniques are used for solving (1) such as spectral collocation method (Ezz-Eldien & Doha, 2019; Ghoreishi & Mokhtary, 2014; Ishtiaq et al., 2009a,b; Wei & Chen, 2012; Yuzbasi, 2014), iterated collocation method Brunner (2004), piecewise collocation method (Ishitawa & Muroya, 2009; Muroya et al., 2003), Legendre Galerkin method (Alsuyuti et al., 2019; Cai & Qi, 2016; Mokhtary, 2019), computational Tau scheme (Ansari & Mokhtary, 2019; Mokhtary, 2014, 2016, 2017) and etc. But all aforementioned methods have been applied to the linear case and there are a few numerical methods to obtain approximate solutions of nonlinear Volterra integro-differential equations with proportional delay in the literature.

In Mokhtary (2019) computational Jacobi Galerkin method for multiple delay pantograph integral equations have been investigated by the author. In this procedure a new well-posed approach for pantograph integral equations is given with convergence analysis. Numerical results confirm the theoretical predictions of the well-posedness and spectral accuracy. The author also provided a camparison of condition numbers and accuracy between this scheme and the spectral collocation method proposed in Ishtiaq et al. (2009a) where we can see a meaningful growth in errors and condition numbers while the errors of the Legendre Galerkin algorithm are decreased versus N and condition numbers remain bounded. We considered applicable and useful shifted Jacobi polynomials (Alsuyuti et al., 2019; Doha et al., 2011; Ezz-Eldien & Doha, 2019; Ezz-Eldien et al., 2017) and Ansari & Mokhtary (2019); Mokhtary (2019) then we extend these computational schemes for the numerically solving nonlinear Volterra integro-differential equations with proportional delay.

In this paper, we develop and analyze a high-order operational Jacobi Tau scheme for the numerical solution of (1). The main idea of the Jacobi Tau scheme is to express the solution of the problem as a finite sum of given basis of functions and converts nonlinear VIDE's with proportional delay to a system of nonlinear algebraic equations which can be solved directly by forward substitution method.

The outlines of the present paper are arranged as follows. In Section 2, the numerical scheme of the considered equation is explained. Section 3, is devoted to the illustrative results. Finally, Section 4 outlines the conclusions.

2 Numerical scheme

The main purpose of this section is to give a computational Jacobi Tau scheme for the numerical solution of (1). To this end, we consider

$$y_N(x) = \sum_{i=0}^{N} y_i J_i^{\alpha,\beta}(x) = \underline{y}\ \underline{\mathbf{J}}, \tag{2}$$

as the Tau approximation of (1) where $\underline{y} = [y_0, y_1, ..., y_N, 0, ...]$ and

$$\underline{\mathbf{J}} = \mathbf{J}\underline{X} = [J_0^{\alpha,\beta}(x), J_1^{\alpha,\beta}(x), ..., J_N^{\alpha,\beta}(x), ...]^T,$$

is the shifted Jacobi polynomial bases on Ω (Canuto et al., 2007; Guo, 1998; Hesthaven et al., 2007; Shen et al., 2011). Here $\mathbf{J}$ is an infinite order upper triangular coefficient matrix with degree $J_i^{\alpha,\beta}(x) \leq i$ for $i = 0, 1, 2, ...,$ and $\underline{X} = [1, x, x^2, ..., x^N, ...]^T$. Now we obtain matrix and vector multiplications for differential term of equation (1), to this end following the operational Tau method which proposed by Ortiz & Samara (1981) and other related papers in

Hossieni & Shahmorad (2007); Pour-Mahmoud et al. (2005); Shahmorad (2005) are based on three simple matrices

$$\mu = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad \ell = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & \frac{1}{2} & 0 & \cdots \\ 0 & 0 & 0 & \frac{1}{3} & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

$$\eta = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 0 & 0 & \cdots \\ 0 & 0 & 3 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Lemma 1. *(Pour-Mahmoud et al., 2005) Let $y(x)$ be a polynomial as*

$$y(x) = \sum_{i=0}^{\infty} y_i J_i^{\alpha,\beta}(x) \simeq y_n \underline{\boldsymbol{J}},$$

then we have

$$\begin{array}{ll} (i) & \dfrac{\partial^r y(x)}{\partial x^r} = y_n \boldsymbol{J} \eta^r \underline{X}, \quad r = 1, 2, 3, \ldots, \\ (ii) & x^r y(x) = y_n \boldsymbol{J} \mu^r \underline{X}, \quad r = 1, 2, 3, \ldots, \\ (iii) & \displaystyle\int_0^x y(s) ds = y_n \boldsymbol{J} \ell \underline{X} - y_n \boldsymbol{J} \ell A, \end{array}$$

where $y_n = [y_0, y_1, \ldots, y_n, 0, 0, \ldots], A = [1, 0, 0, \ldots, 0, 0, \ldots]^T$.

For any linear differential operator $\mathcal{D}$ defined in equation (1) and any series

$$y(x) = \sum_{i=0}^{\infty} y_i J_i^{\alpha,\beta}(x) = \underline{y}\ \underline{\mathbf{J}},$$

we have

$$\mathcal{D}y(x) = \underline{y} \Pi_{\mathbf{J}} \underline{\mathbf{J}} = \underline{y} \Pi_{\mathbf{J}} \mathbf{J} \underline{X}, \tag{3}$$

where $\Pi = \sum\limits_{i=0}^{n_d} \eta^i p_i(\mu), \quad \Pi_{\mathbf{J}} = \mathbf{J} \Pi \mathbf{J}^{-1}$. Assume $d = [d_0, d_1, \ldots, d_{\xi-1}, 0, 0, \ldots]$ be the vector that contains right hand sides of conditions. Now we give matrix form for initial conditions, then by using part (i) of Lemma 1, we can write:

$$y^{(\zeta)}(0) = \underline{y} \mathbf{J} \eta^\zeta \underline{X}\Big|_{x=0} = \underline{y} \mathbf{J} \eta^\zeta e_1 = \underline{y} b_\zeta, \quad \zeta = 0, 1, \ldots, \xi - 1, \tag{4}$$

where $X\Big|_{x=0} = e_1 = [1, 0, 0, \ldots]^T$ is the first column of identity matrix. We definite columns of matrix B^* as

$$B^* = \Big[b_\zeta\Big]_{\zeta=0}^{\xi-1} = \Big[\mathbf{J} \eta^\zeta e_1\Big]_{\zeta=0}^{\xi-1},$$

so we have

$$\underline{y} B^* = d. \tag{5}$$

Assume that

$$f(x)=\sum_{i=0}^{\infty} f_i J_i^{\alpha,\beta}(x)=\underline{f}\ \mathbf{J},\quad \underline{f}=[f_0,f_1,...],$$
$$F(t,y(t))\simeq\sum_{s=0}^{N}\rho_s(t)J_s^{\alpha,\beta}(y(t))=\sum_{s=0}^{N}\tilde{\rho}_s(t)y^s(t).$$

Substituting (2),(3),(4),(5) and above relations into (1) concludes

$$\begin{aligned}\underline{y}\ \Pi_{\mathbf{J}}\underline{\mathbf{J}} &= \underline{f}\ \underline{\mathbf{J}}+\sum_{s=0}^{N}\int_0^{qx}K(x,t)\tilde{\rho}_s(t)\big(\underline{y}\mathbf{J}\underline{X}^t\big)^s dt\\ &= \underline{f}\ \underline{\mathbf{J}}+\int_0^{qx}K(x,t)\tilde{\rho}_0(t)dt+\sum_{s=1}^{N}\int_0^{qx}K_s(x,t)\big(\underline{y}\mathbf{J}\underline{X}^t\big)^s dt, \end{aligned}\tag{6}$$

with $K_s(x,t)=K(x,t)\tilde{\rho}_s(t)$ for $s=1,2,...,N$ and $\underline{X}^t=[1,t,t^2,...,t^N,...]^T$.

From Ghoreishi & Hadizadeh (2009), we can find that the matrix formulation of $\big(\underline{y}\mathbf{J}\underline{X}^t\big)^s$ as follows

$$\big(\underline{y}\mathbf{J}\underline{X}^t\big)^s=\underline{y}\mathbf{J}\Upsilon^{s-1}\underline{X}^t,\quad s=1,2,...,N,\tag{7}$$

where Υ has the following infinite order upper triangular toeplitz matrix form

$$\Upsilon=\begin{pmatrix}\tilde{y}_0 & \tilde{y}_1 & \tilde{y}_2 & \cdots\\ 0 & \tilde{y}_0 & \tilde{y}_1 & \cdots\\ 0 & 0 & \tilde{y}_0 & \cdots\\ \vdots & \vdots & \vdots & \ddots\end{pmatrix},\tag{8}$$

with $\underline{y}\mathbf{J}=\widetilde{y}=[\tilde{y}_0,\tilde{y}_1,...,\tilde{y}_N,0,...]$. Substituting (7) into (6) yields

$$\underline{y}\left(\Pi_{\mathbf{J}}\underline{\mathbf{J}}\ \mathbf{J}^{-1}-\sum_{s=1}^{N}\Upsilon^{s-1}\int_0^{qx}K_s(x,t)\underline{X}^t dt\right)\mathbf{J}$$
$$=\underline{f}\ \underline{\mathbf{J}}+\int_0^{qx}K(x,t)\tilde{\rho}_0(t)dt.\tag{9}$$

Assuming

$$\begin{aligned}K(x,t) &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}k_{i,j}J_i^{\alpha,\beta}(x)J_j^{\alpha,\beta}(t)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\tilde{k}_{i,j}x^i t^j\\ K_s(x,t) &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}k^s_{i,j}J_i^{\alpha,\beta}(x)J_j^{\alpha,\beta}(t)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\tilde{k}^s_{i,j}x^i t^j,\quad s=1,2,...,N_F,\\ \tilde{\rho}_0(t) &= \sum_{i=0}^{\infty}\tilde{\rho}_{i,0}J_i^{\alpha,\beta}(t)=\underline{\tilde{\rho}}\mathbf{J}\underline{X}^t,\quad \underline{\tilde{\rho}}=[\tilde{\rho}_{0,0},\tilde{\rho}_{1,0},...],\end{aligned}$$

where $\tilde{k}_{i,j}$ and $\tilde{k}^s_{i,j}$ are obtained by rearranging the orthogonal expansions of $K(x,t)$ and $K_s(x,t)$

based on the powers of $x^i t^j$ respectively, so can rewrite equation (9) as follows

$$\underline{y}\Bigg(\Pi_{\mathbf{J}}\ \underline{X} - \sum_{s=1}^{N}\Upsilon^{s-1}\Bigg(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\tilde{k}^{s}_{i,j}x^i\int\limits_0^{qx} t^j\underline{X}^t dt\Bigg)\Bigg)\mathbf{J}$$

$$= \underline{f}\ \underline{\mathbf{J}} + \underline{\tilde{\rho}}\mathbf{J}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\tilde{k}_{i,j}x^i\int\limits_0^{qx} t^j\underline{X}^t dt. \tag{10}$$

On the other hand, following the relation

$$\int\limits_0^{qx} t^j\underline{X}^t dt = \left[\int\limits_0^{qx} t^{j+m}dt\right]_{m=0}^{\infty} = \left[\frac{(qx)^{j+m+1}}{j+m+1}\right]_{m=0}^{\infty},$$

we can write

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\tilde{k}^{s}_{i,j}x^i\int\limits_0^{qx} t^j\underline{X}^t dt$$

$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\tilde{k}^{s}_{i,j}\left[\frac{q^{j+m+1}}{j+m+1}x^{i+j+m+1}\right]_{m=0}^{\infty} = \mathcal{K}_s\underline{X}, \tag{11}$$

where $\mathcal{K}_s$ is the infinite order upper triangular matrix with only nonzero entries

$$(\mathcal{K}_s)_{m,m+r+1} = \sum_{i=0}^{r}\frac{\tilde{k}^{s}_{i,r-i}}{r-i+m+1}q^{r-i+m+1}, \quad m,r = 0,1,\ldots\,. \tag{12}$$

Similarly, we can write

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\tilde{k}_{i,j}x^i\int\limits_0^{qx} t^j\underline{X}^t dt = \mathcal{K}\underline{X}, \tag{13}$$

where $\mathcal{K}$ is the infinite order upper triangular matrix with only nonzero entries that are given by inserting $\tilde{k}_{i,r-i}$ instead of $\tilde{k}^{s}_{i,r-i}$ in (12).

Substituting the relations (11) and (13) into (10) we obtain

$$\underline{y}\Bigg(\mathbf{J}\Big(\sum_{i=0}^{n_d}\eta_i p_i(\mu)\Big) - \sum_{s=1}^{N}\Upsilon^{s-1}\mathcal{K}_s\mathbf{J}\Bigg)\mathbf{J}^{-1}\underline{\mathbf{J}} = \Big(\underline{f}\mathbf{J} + \underline{\tilde{\rho}}\mathbf{J}\mathcal{K}\Big)\mathbf{J}^{-1}\underline{\mathbf{J}}. \tag{14}$$

To complete the Jacobi Tau discretization of (1), we first project (14) on $\{J_i^{\alpha,\beta}(x)\}_{i=0}^{N}$ and then multiply both sides of the equation obtained on $\mathbf{J}$. Thus we obtain

$$\underline{\tilde{y}}_N\Gamma_N = \underline{\tilde{f}}_N, \tag{15}$$

when $\underline{\tilde{y}}_N = [\tilde{y}_0, \tilde{y}_1, ..., \tilde{y}_N]$ and $\underline{\tilde{f}}_N = [\tilde{f}_0, \tilde{f}_1, ..., \tilde{f}_N]$ are the first $N+1$ entries of the infinite vectors $\underline{\tilde{y}} = \underline{y}\mathbf{J}$ and $\underline{f}\mathbf{J}$ respectively. Moreover Γ_N is the principle submatrix of order $N+1$ from the infinite matrix $\left(\sum\limits_{i=0}^{n_d}\eta_i p_i(\mu) - \sum\limits_{s=1}^{N}\Upsilon^{s-1}\mathcal{K}_s - \underline{\tilde{\rho}}\mathbf{J}\mathcal{K}\right)$.

Now, we focus on the uniquely solvability of the nonlinear algebraic system (15) and give a computational algorithm to solve it. To this end we investigate the structure of the matrix $\left(\sum\limits_{i=0}^{n_d}\eta_i p_i(\mu) - \sum\limits_{s=1}^{N}\Upsilon^{s-1}\mathcal{K}_s - \underline{\tilde{\rho}}\mathbf{J}\mathcal{K}\right)$. In Refs. Ansari & Mokhtary (2019); Ghoreishi & Hadizadeh

(2009) the structure of the $\Upsilon^{s-1}, \mathcal{K}_s, \sum_{s=1}^{N} \Upsilon^{s-1}\mathcal{K}_s$ are given by the upper triangular toeplitz matrix forms. Thus we can conclude

$$
\begin{aligned}
&\underline{y}\Big(\sum_{s=1}^{N} \Upsilon^{s-1}\mathcal{K}_s\mathbf{J}\Big) + \underline{\tilde{\rho}}\mathbf{J}\mathcal{K} \\
&\qquad = \Big[0, \Lambda_0(\tilde{y}_0), \Lambda_1(\tilde{y}_0, \tilde{y}_1), ..., \Lambda_{N-1}(\tilde{y}_0, \tilde{y}_1, ..., \tilde{y}_{N-1}), ...\Big],
\end{aligned}
\tag{16}
$$

$$
\underline{\tilde{y}}^* := \underline{y}\ \mathbf{J}\Big(\sum_{i=0}^{n_d} \eta_i p_i(\mu)\Big) = \underline{\tilde{y}}\Big(\sum_{i=0}^{n_d} \eta_i p_i(\mu)\Big),
$$

where $\Lambda_i(\tilde{y}_0, \tilde{y}_1, ..., \tilde{y}_i),\ \ i \geq 0$ are the nonlinear functions with elements $\tilde{y}_0, \tilde{y}_1, ..., \tilde{y}_i$. Using (16) and (5) the nonlinear algebraic system (15) can be rewritten as

$$
\begin{cases}
\underline{\tilde{y}}\ \Gamma_j = \underline{\tilde{f}}_j, \quad j = 0, 1, \ldots, N, \\
\underline{y}\ B^* = d,
\end{cases}
\tag{17}
$$

we can rewrite system (17) as follows

$$
\begin{aligned}
y_0 B_0^* &= d_0 \\
&\ \vdots \\
y_{\xi-1} B_{\xi-1}^* &= d_{\xi-1} \\
\underline{\tilde{y}}_0^* &= \underline{\tilde{f}}_0 \\
\underline{\tilde{y}}_1^* &= \underline{\tilde{f}}_1 + \Lambda_0(\tilde{y}_0) \\
\underline{\tilde{y}}_2^* &= \underline{\tilde{f}}_2 + \Lambda_1(\tilde{y}_0, \tilde{y}_1) \\
&\ \vdots \\
\underline{\tilde{y}}_N^* &= \underline{\tilde{f}}_N + \Lambda_{N-1}(\tilde{y}_0, \tilde{y}_1, ..., \tilde{y}_{N-1}),
\end{aligned}
\tag{18}
$$

which can be solved exactly using the forward substitution method. Finally the computational Jacobi Tau approximation of (1) can be obtained by solving the triangular system $\underline{y}\mathbf{J} = \underline{\tilde{y}}$ and $\underline{\tilde{y}}^*$ using the backward substitution method and substituting obtained $\underline{y}$ into (2). Following the Refs. Ansari & Mokhtary (2019); Atkinson (1997); Atkinson & Han (2009), the proofs of convergence and stability analysis can be analyze and investigate, we leave the details of the proofs to the reader.

3 Numerical results

In this section, we listed several examples to illustrate the powerful and effectiveness of the operational Jacobi Tau scheme for the numerical solution of (1). To this end, we require some definitions as follows.

- (Shen et al., 2011)$\|.\|_\infty$ is the uniform norm defined by $\|y\|_\infty = \max_{x\in\Omega} |y(x)|$.
- $L^2(\Omega)$ is the space of functions whose square is integrable in Ω with the norm

$$
\Big\|y\Big\|_2^2 = (y, y) := \int_\Omega y^2(x)dx,
$$

where $(.,.)$ is the inner product formula.

All the numerical computations have been done using Matlab software. In the tables presented here, "Numerical errors" always refer to the error functions in $L^1(\Omega), L^2(\Omega), L^\infty(\Omega)$ respectively, i.e., $\|e_N\|_1, \|e_N\|_2, \|e_N\|_\infty$ and "order$_N$" always refers to the order of convergence which is calculated by,

$$\text{order}_N = \left|\log_2 \frac{\|e_N\|_2}{\|e_{\frac{N}{2}}\|_2}\right|.$$

All the reported results approve the effectiveness of the proposed computational scheme.

Example 1. *Consider the following problem*

$$\begin{cases} \mathcal{D}y(x) = f(x) + \int\limits_0^{qx} (1+xt)y^2(t)dt,, \quad q \in (0,1),\ x \in \Omega = [0,1], \\ y^{(\zeta)}(0) = 1, \quad \zeta = 0,1, \end{cases}$$

where $q = 0.05$, $f(x) = 1 + \frac{19x}{20} - \frac{x^2}{400} - \frac{31x^3}{24000} - \frac{x^4}{12000} - \frac{x^5}{640000}$, *and* $y(x) = 1 + x$. *For linear differential operator we have*

$$\mathcal{D} := \sum_{i=0}^{1} p_i(x) \frac{\partial^i}{\partial x^i},$$

where $p_0(x) = 0, p_1(x) = 1 + x$.

We explain the algorithm details for $N = 5$ and so on the unknown coefficients of the approximate solution (2) satisfy in the nonlinear algebraic system (18) where the nonlinear functions $\Lambda_i(\tilde{y}_0, \tilde{y}_1, ..., \tilde{y}_i)$ for $i = 0, 1, ..., 4$ are obtained from (16). Since $F(t, y(t)) = y^2(t)$ then (16) has the following form

$$\underline{y}\mathbf{J}\left(\sum_{s=0}^{N} \Upsilon^{s-1}\mathcal{K}_s\right) = \underline{y}\mathbf{J}\Upsilon\mathcal{K}_2, \tag{19}$$

where the matrix Υ defined in (8) and by using of shifted Legendre polynomial coefficients

$$\mathbf{J} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 \\ 1 & -6 & 6 & 0 & 0 & 0 \\ -1 & 12 & -30 & 20 & 0 & 0 \\ 1 & -20 & 90 & -140 & 70 & 0 \\ -1 & 30 & -210 & 560 & -630 & 252 \end{pmatrix}.$$

In addition $\Pi, \mathcal{K}_2$ have the following forms which are obtained from (3) , (12) respectively

$$\Pi = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 & 4 & 0 \\ 0 & 0 & 0 & 0 & 5 & 5 \end{pmatrix}, \quad \mathcal{K}_2 = \begin{pmatrix} 0 & q & 0 & \frac{q^2}{2} & 0 & 0 \\ 0 & 0 & \frac{q^2}{2} & 0 & \frac{q^3}{3} & 0 \\ 0 & 0 & 0 & \frac{q^3}{3} & 0 & \frac{q^4}{4} \\ 0 & 0 & 0 & 0 & \frac{q^4}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{q^5}{5} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus (19) and (16) concludes

$$\begin{aligned} &\Lambda_0(\tilde{y}_0) = q\tilde{y}_0, \\ &\Lambda_1(\tilde{y}_0, \tilde{y}_1) = q^2\tilde{y}_0\tilde{y}_1, \\ &\Lambda_2(\tilde{y}_0, \tilde{y}_1, \tilde{y}_2) = \frac{q^3}{3}(\tilde{y}_1^2 + 2\tilde{y}_0\tilde{y}_2) + \frac{q^2}{2}\tilde{y}_0^2, \\ &\Lambda_3(\tilde{y}_0, \tilde{y}_1, \tilde{y}_2, \tilde{y}_3) = \frac{q^4}{4}(2\tilde{y}_0\tilde{y}_3 + 2\tilde{y}_1\tilde{y}_2) + \frac{q^3}{3}(2\tilde{y}_0\tilde{y}_1), \\ &\Lambda_4(\tilde{y}_0, \tilde{y}_1, \tilde{y}_2, \tilde{y}_3, \tilde{y}_4) = \frac{q^4}{4}(\tilde{y}_1^2 + 2\tilde{y}_0\tilde{y}_2) + \frac{q^5}{5}(\tilde{y}_2^2 + 2\tilde{y}_0\tilde{y}_4 + 2\tilde{y}_1\tilde{y}_3). \end{aligned} \tag{20}$$

On the other hand by using the initial conditions and (4) yields

$$\begin{cases} \tilde{y}_0 = 1, \\ \tilde{y}_1 = 1. \end{cases} \tag{21}$$

Inserting (20) into (18) and using

$$\underline{\widetilde{f}}_5 = [\underline{\widetilde{f}}_0, \underline{\widetilde{f}}_1, ..., \underline{\widetilde{f}}_5] = [1, \frac{19}{20}, -\frac{1}{400}, -\frac{31}{24000}, -\frac{1}{12000}, -\frac{1}{640000}],$$

we find the unknowns coefficients as follows

$$\begin{aligned}
&\underline{\widetilde{y}}_0^* = \tilde{y}_1 = \underline{\widetilde{f}}_0 = 1, \\
&\underline{\widetilde{y}}_1^* = \tilde{y}_1 + 2\tilde{y}_2 = \underline{\widetilde{f}}_1 + \Lambda_0(\tilde{y}_0) = \underline{\widetilde{f}}_1 + q\tilde{y}_0, \\
&\underline{\widetilde{y}}_2^* = 2\tilde{y}_2 + 3\tilde{y}_3 = \underline{\widetilde{f}}_2 + \Lambda_1(\tilde{y}_0, \tilde{y}_1) = \underline{\widetilde{f}}_2 + q^2\tilde{y}_0\tilde{y}_1, \\
&\underline{\widetilde{y}}_3^* = 3\tilde{y}_3 + 4\tilde{y}_4 = \underline{\widetilde{f}}_3 + \Lambda_2(\tilde{y}_0, \tilde{y}_1, \tilde{y}_2) = \underline{\widetilde{f}}_3 + \frac{q^3}{3}(\tilde{y}_1^2 + 2\tilde{y}_0\tilde{y}_2) + \frac{q^2}{2}\tilde{y}_0^2, \\
&\underline{\widetilde{y}}_4^* = 4\tilde{y}_4 + 5\tilde{y}_5 = \underline{\widetilde{f}}_4 + \Lambda_3(\tilde{y}_0, \tilde{y}_1, \tilde{y}_2, \tilde{y}_3) \\
&= \underline{\widetilde{f}}_4 + \frac{q^4}{4}(2\tilde{y}_0\tilde{y}_3 + 2\tilde{y}_1\tilde{y}_2) + \frac{q^3}{3}(2\tilde{y}_0\tilde{y}_1), \\
&\underline{\widetilde{y}}_5^* = 5\tilde{y}_5 = \underline{\widetilde{f}}_5 + \Lambda_4(\tilde{y}_0, \tilde{y}_1, \tilde{y}_2, \tilde{y}_3, \tilde{y}_4) \\
&= \underline{\widetilde{f}}_5 + \frac{q^4}{4}(\tilde{y}_1^2 + 2\tilde{y}_0\tilde{y}_2) + \frac{q^5}{5}(\tilde{y}_2^2 + 2\tilde{y}_0\tilde{y}_4 + 2\tilde{y}_1\tilde{y}_3).
\end{aligned} \tag{22}$$

Finally, by solving system of equations (21) , (22) and inserting solution into (2) the Legendre Tau solution of the problem is given by

$$y_5(x) = \underline{y}\mathbf{J}\underline{X} = \underline{\widetilde{y}}\ \underline{X} = [1, 1, 0, 0, ...]\underline{X} = 1 + x,$$

which is the exact solution.

Example 2. *Consider the following problem*

$$\begin{cases} \mathcal{D}y(x) = f(x) + \int\limits_0^{qx} 3\sin(x-t)\cos^2(t)dt, \\ y(0) = 1, \quad y^{'}(0) = 0, \quad y^{''}(0) = 1, \end{cases}$$

where $n_d = 1, p_0(x) = 1, p_1(x) = 0$ *and we choose* $f(x)$ *with respect to* q *such as to be the exact solution* $y(x) = \cos x$.

From the reported results in Table 1 and Figure 1, we observed that a good approximation of the nonlinear VIDE's with proportional delay is achieved for small values of N. *The presented errors in Fig. 1 and order variations in Table 1, approve the predicted exponential-like (infinite order) rate of convergence for the proposed method because the orders tend to infinity versus* N *and the errors representation is linear. This confirms the reliability and well-posedness of the proposed scheme in obtaining the approximate solution of the problem.*

Example 3. *Considering the following problem*

$$\begin{cases} \mathcal{D}y(x) = f(x) + \int\limits_0^{qx} (e^t)^3 dt, \\ y^{(\zeta)}(0) = 1, \quad \zeta = 0, 1, 2, \end{cases}$$

where $n_d = 2, p_0(x) = 2, p_1(x) = -1$ *and* $y(x) = e^x$ *is the exact solution.*

Table 1: Approximation errors in various norms for example 2 with using shifted Legendre polynomials ($\alpha = \beta = 0$), $q = 0.05$.

N	$\|e_N\|_1$	$\|e_N\|_2$	$\|e_N\|_\infty$	$Order_N$
2	8.1377×10^{-3}	1.3516×10^{-2}	4.0302×10^{-2}	–
4	1.9568×10^{-4}	3.7931×10^{-4}	1.3644×10^{-3}	3.5733
6	2.7308×10^{-6}	5.9559×10^{-6}	2.4528×10^{-5}	7.7272
8	2.4892×10^{-8}	5.9721×10^{-8}	2.7350×10^{-7}	8.7564
10	1.5983×10^{-10}	4.1542×10^{-10}	2.0763×10^{-9}	13.7246
12	7.6191×10^{-13}	2.1218×10^{-12}	1.1423×10^{-11}	14.8476
14	2.8029×10^{-15}	8.2939×10^{-15}	4.7637×10^{-14}	20.3921
16	8.5815×10^{-18}	2.6131×10^{-17}	1.5774×10^{-16}	21.5498
18	3.6083×10^{-19}	5.9285×10^{-19}	1.5515×10^{-18}	25.3358
20	3.8041×10^{-19}	6.4138×10^{-19}	1.9626×10^{-18}	–

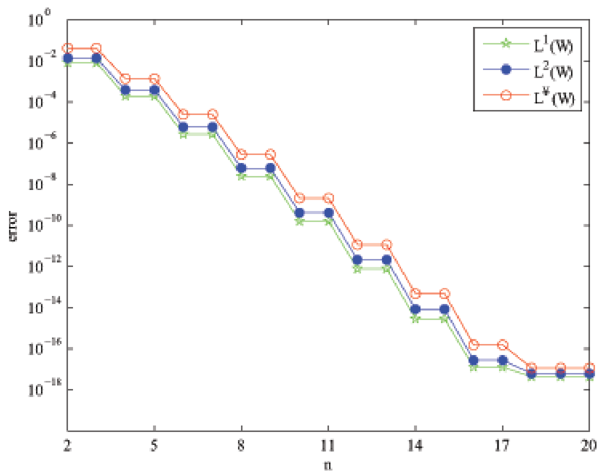

Figure 1: We observe approximation errors in various norms for example 2 with using shifted Chebyshev polynomials($\alpha = \beta = -\frac{1}{2}$), $q = 0.95$.

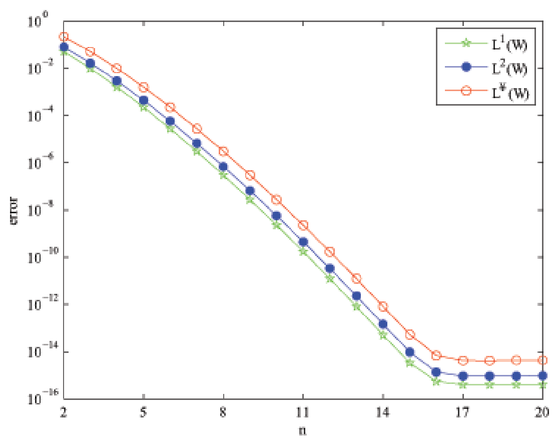

Figure 2: We observe approximation errors in various norms for example 3 with using shifted Chebyshev polynomials, $q = 0.95$.

Table 2: Approximation errors in various norms for example 3 with using shifted Legendre polynomials, $q = 0.05$.

N	$\|e_N\|_1$	$\|e_N\|_2$	$\|e_N\|_\infty$	$Order_N$
2	5.1615×10^{-2}	7.9682×10^{-2}	2.1828×10^{-1}	–
4	1.6152×10^{-3}	2.9539×10^{-3}	9.9485×10^{-3}	3.2949
6	2.7860×10^{-5}	5.7928×10^{-5}	2.2627×10^{-4}	5.6715
8	3.0289×10^{-7}	6.9792×10^{-7}	3.0586×10^{-6}	8.3503
10	2.2606×10^{-9}	5.6740×10^{-9}	2.7313×10^{-8}	11.2654
12	1.2286×10^{-11}	3.3180×10^{-11}	1.7288×10^{-10}	14.3728
14	5.0771×10^{-14}	1.4616×10^{-13}	8.1549×10^{-13}	17.6424
16	1.6482×10^{-16}	5.0224×10^{-16}	2.9762×10^{-15}	21.0523
18	4.0848×10^{-19}	1.3567×10^{-18}	8.5508×10^{-18}	24.6049
20	2.2130×10^{-20}	3.4427×10^{-20}	8.1559×10^{-20}	25.8281

Example 4. *Assume that*

$$\begin{cases} \mathcal{D}y(x) = f(x) + \int\limits_0^{qx} x^3 \cos(t) e^{\sin(t)} dt, \\ y^{(\zeta)}(0) = 0, \quad y^{(\zeta+1)}(0) = 1, \quad \zeta = 0, 2, \end{cases}$$

where $n_d = 3, p_0(x) = 1, p_1(x) = 2, p_2(x) = 0, p_3(x) = -2$ *and the exact solution is given by* $y(x) = \sin x$.

Table 3: Approximation errors in various norms for example 4 with using shifted Legendre polynomials, $q = 0.05$.

N	$\|e_N\|_1$	$\|e_N\|_2$	$\|e_N\|_\infty$	$Order_N$
6	2.4528×10^{-5}	5.0607×10^{-5}	1.9568×10^{-4}	–
8	2.7350×10^{-7}	6.2704×10^{-7}	2.7308×10^{-6}	–
10	2.0763×10^{-9}	5.1931×10^{-9}	2.4892×10^{-8}	–
12	1.1423×10^{-11}	3.0769×10^{-11}	1.5983×10^{-10}	14.3131
14	4.7639×10^{-14}	1.3687×10^{-13}	7.6191×10^{-13}	15.3375
16	1.5578×10^{-16}	4.7391×10^{-16}	2.8033×10^{-15}	21.0033

Example 5. *Consider the following problem*

$$\begin{cases} \mathcal{D}y(x) = f(x) + \int\limits_0^{qx} e^x \cos^2(t) \tan^2(t) dt, \\ y^{(\zeta)}(0) = 0, \quad y^{(\zeta+1)}(0) = 1, \quad \zeta = 0, 2, \end{cases}$$

where $n_d = 2, p_0(x) = 1, p_1(x) = 2\tan x, p_2(x) = -1$ *and the exact solution of equation is* $y(x) = \tan x$.

Example 6. *Assuming that*

$$\begin{cases} \mathcal{D}y(x) = f(x) + \int\limits_0^{qx} x e^t \sin(e^t) dt, \\ y^{(\zeta)}(0) = 1, \quad \zeta = 0, 1, \end{cases}$$

where $n_d = 1, p_0(x) = 0, p_1(x) = 1$ *and* $y(x) = e^x$.

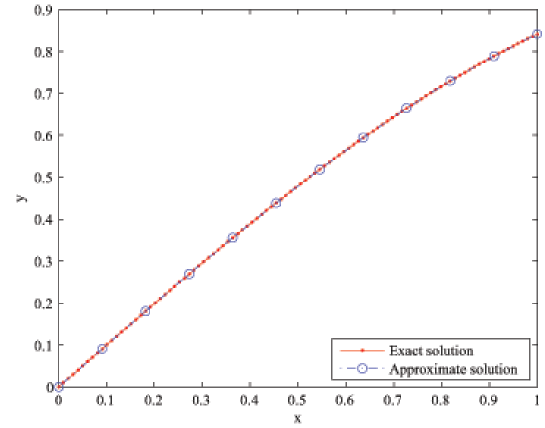

Figure 3: We observe the exact and approximate solutions for example 4 with using shifted Legendre polynomials, $N = 5, q = 0.05$.

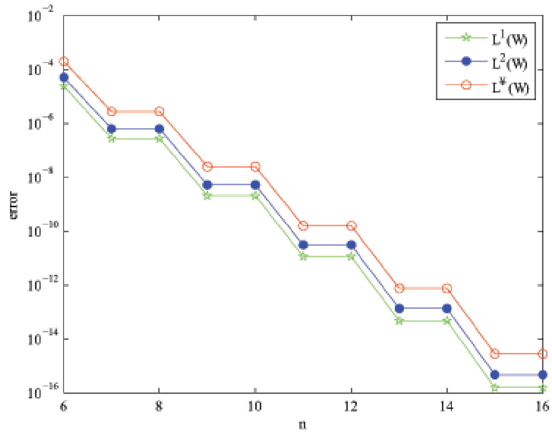

Figure 4: We observe approximation errors in various norms for example 4 with using shifted Chebyshev polynomials, $q = 0.95$.

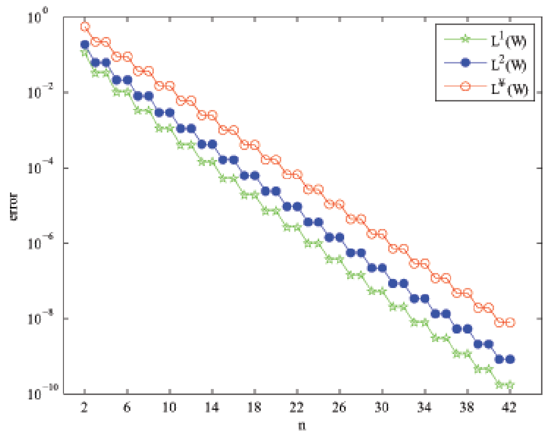

Figure 5: We observe approximation errors in various norms for example 5 with using shifted Chebyshev polynomials, $q = 0.95$.

Given results implies that the errors decay as the degree of approximation N tends to infinity. Furthermore, in this procedure we observe the linear diagram of the errors representations in all

Table 4: Approximation errors in various norms for example 5 with using shifted Legendre polynomials, $q = 0.05$.

N	$\|e_N\|_1$	$\|e_N\|_2$	$\|e_N\|_\infty$	$Order_N$
2	1.1563×10^{-1}	1.8592×10^{-1}	5.5741×10^{-1}	–
6	1.0071×10^{-2}	2.1818×10^{-2}	9.0741×10^{-2}	1.0395
10	1.1379×10^{-3}	2.9561×10^{-3}	1.4903×10^{-2}	1.9989
14	1.4275×10^{-4}	4.2305×10^{-4}	2.4480×10^{-3}	2.9341
18	1.8982×10^{-5}	6.2397×10^{-5}	4.0209×10^{-4}	3.8581
22	2.6200×10^{-6}	9.3820×10^{-6}	6.6046×10^{-5}	4.7760
26	3.7118×10^{-7}	1.4297×10^{-6}	1.0848×10^{-5}	5.6900
30	5.3605×10^{-8}	2.2001×10^{-7}	1.7819×10^{-6}	6.6016
34	7.8568×10^{-9}	3.4115×10^{-8}	2.9269×10^{-7}	7.5115
38	1.1651×10^{-9}	5.3214×10^{-9}	4.8076×10^{-8}	8.4202
42	1.7443×10^{-10}	8.3411×10^{-10}	7.8967×10^{-9}	9.3279

Table 5: Approximation errors in various norms for example 6 with using shifted Legendre polynomials, $q = 0.05$.

N	$\|e_N\|_1$	$\|e_N\|_2$	$\|e_N\|_\infty$	$Order_N$
2	5.1612×10^{-2}	7.9678×10^{-2}	2.1827×10^{-1}	–
4	1.6152×10^{-3}	2.9539×10^{-3}	9.9485×10^{-3}	3.2949
6	2.7860×10^{-5}	5.7928×10^{-5}	2.2627×10^{-4}	5.6715
8	3.0289×10^{-7}	6.9792×10^{-7}	3.0586×10^{-6}	8.3505
10	2.2606×10^{-9}	5.6740×10^{-9}	2.7313×10^{-8}	11.2654
12	1.2286×10^{-11}	3.3180×10^{-11}	1.7288×10^{-10}	14.3728
14	5.0766×10^{-14}	1.4616×10^{-13}	8.1547×10^{-13}	17.6425

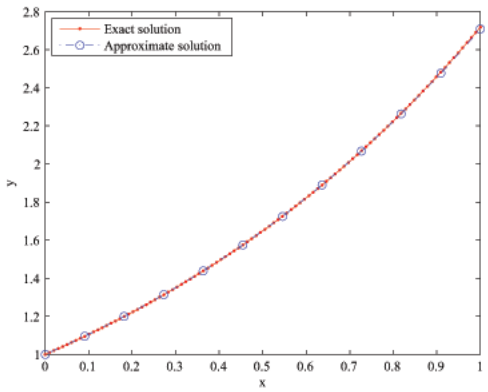

Figure 6: We observe the exact and approximate solutions for example 6 with using shifted Chebyshev polynomials, $N = 4, q = 0.05$.

figures and moving the orders in all tables towards infinity versus N which approved the predicted exponentially rate of convergence. These properties confirm that our proposed scheme produces a high order of accuracy approximate solution for (1).

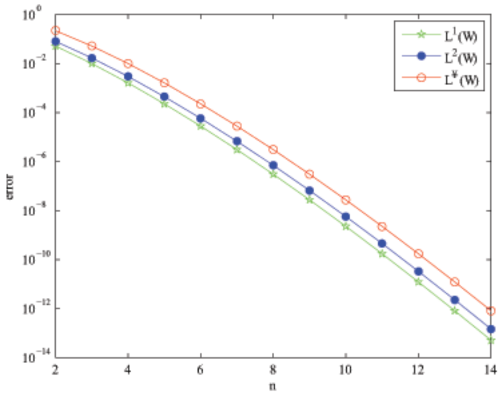

Figure 7: We observe approximation errors in various norms for example 6 with using shifted Chebyshev polynomials, $q = 0.95$.

4 Conclusion

In this paper, we presented a computational and highly accurate operational Jacobi Tau scheme for the numerical solution of the nonlinear Volterra integro-differential equations with proportional delay. Our motivation in this procedure is to convert nonlinear Volterra integro-differential equations with proportional delay to a system of nonlinear algebraic equations, which can be solved directly by forward substitution method. From the tables, numerical examples were given to confirm the effectiveness and reliability of the our scheme. The authors are grateful to the referees for their useful suggestions and comments on this paper which made it complete.

References

Alsuyuti, M.M., Doha, E.H., Ezz-Eldien, S.S., Bayoumi, B.I. & Baleanu, D. (2019). Modified Galerkin algorithm for solving multitype fractional differential equations. *Mathematical Methods in the Applied Sciences, 42*(5).

Ansari, H. & Mokhtary, P. (2019). Computational Legendre Tau method for Volterra Hammerstein pantograph integral equations. *Bulletin of the Iranian Mathematical Society, 45*(2), 475-493.

Atkinson, K.E. (1997). *The Numerical Solution of Integral Equations of the Second Kind*. Cambridge.

Han, W. & Atkinson, K.E. (2009). *Theoretical Numerical Analysis: A Functional Analysis Framework*. Springer New York.

Bellen, A. & Zennaro, M. (2003). *Numerical Methods for Delay Differential Equations*. Oxford University Press, Oxford.

Brunner, H. (2004). *Collocation methods for Volterra integral and related functional differential equations* (Vol. 15). Cambridge university press.

Canuto, C., Hussaini, M.Y., Quarteroni, A. & Zang, T.A. (2007). *Spectral methods: fundamentals in single domains.* Springer Science & Business Media.

Cai, H. & Qi, J. (2016). A Legendre-Galerkin method for solving general Volterra functional integral equations. *Numerical Algorithms, 73*(4), 1159-1180.

Doha, E.H., Bhrawy, A.H. & Ezz-Eldien, S.S. (2011). Efficient Chebyshev spectral methods for solving multi-term fractional orders differential equations. *Applied Mathematical Modelling, 35*(12), 5662-5672.

Ezz-Eldien, S.S. & Doha, E.H. (2019). Fast and precise spectral method for solving pantograph type Volterra integro-differential equations. *Numerical Algorithms, 81*(1), 57-77.

Ezz-Eldien, S.S., Hafez, R.M., Bhrawy, A.H., Baleanu, D. & El-Kalaawy, A.A. (2017). New numerical approach for fractional variational problems using shifted Legendre orthonormal polynomials. *Journal of Optimization Theory and Applications, 174*(1), 295-320.

Ghoreishi, F. & Hadizadeh, M. (2009). Numerical computation of the Tau approximation for the Volterra-Hammerstein integral equations. *Numerical Algorithms, 52*(4), 541-559.

Ghoreishi, F. & Mokhtary, P. (2014). Spectral collocation method for multi-order fractional differential equations. *International Journal of Computational Methods, 11*(05), 1350072.

Guo, B.Y. (1998). *Spectral Methods and Their Applications*, World Scientific Publishing Company, Singapore, River Edge (new Jersey) and London.

Hesthaven, J.S., Gottlieb, S., & Gottlieb, D. (2007). S*pectral methods for time-dependent problems* (Vol. 21). Cambridge University Press.

Hosseini, S.M. & Shahmorad, S. (2003). Tau Numerical Solution Of Fredholm Integro-Differential Equations With Arbitrary Polynomial Bases. *Applied Mathematical Modelling, 27*(2), 145-154.

Iserles, A. & Liu, Y. (1994). On pantograph integro-differential equations. *The Journal of Integral Equations and Applications*, 6, 213-237.

Iserles, A. & Liu, Y. (1997a). Integro-differential equations and generalized hypergeometric functions. *J. Math. Anal. Appl.*, 208, 404–424.

Iserles, A. & Liu, Y. (1997b). On neutral functional-differential equations with proportional delays. *Journal of Mathematical Analysis and Applications, 207*(1), 73-95.

Ishiwata, E. & Muroya, Y. (2009). On collocation methods for delay differential and Volterra integral equations with proportional delay. *Frontiers of Mathematics in China, 4*(1), 89-111.

Ishtiaq, A., Brunner, H. & Tang, T. (2009b). Spectral methods for pantograph-type differential and integral equations with multiple delays. Frontiers of Mathematics in China, 4(1), 49-61.

Ishtiaq, A., Brunner, H. & Tang, T. (2009b). A spectral method for pantograph-type delay differential equations and its convergence analysis. *Journal of Computational Mathematics*, 27, 254-265.

Mokhtary, P. (2014). Operational Tau method for nonlinear multi-order FDEs. *Iranian Journal of Numerical Analysis and Optimization, 4*(2), 43-55.

Mokhtary, P. (2019). Spectrally Accurate and Well-Posed Jacobi Galerkin Method for Multiple Delay Pantograph Integral Equations. *Iranian Journal of Science and Technology, Transactions A: Science, 43*(3), 959-967.

Mokhtary, P. (2016). Numerical treatment of a well-posed Chebyshev Tau method for Bagley-Torvik equation with high-order of accuracy. *Numerical Algorithms, 72*(4), 875-891.

Mokhtary, P. (2017). Numerical analysis of an operational Jacobi Tau method for fractional weakly singular integro-differential equations. *Applied Numerical Mathematics*, 121, 52-67.

Muroya, Y., Ishiwata, E. & Brunner, H. (2003). On the attainable order of collocation methods for pantograph integro-differential equations. *Journal of Computational and Applied Mathematics, 152*(1-2), 347-366.

Ortiz, E.L. & Samara, H.J. (1981). An operational approach to the Tau method for the numerical solution of non-linear differential equations. *Computing, 27*(1), 15-25.

Pour-Mahmoud, J., Rahimi-Ardabili, M.Y. & Shahmorad, S. (2005). Numerical solution of Volterra integro-differential equations by the Tau method with the Chebyshev and Legendre bases. *J. Appl. Math. Comput.*, 170, 314–338.

Shen, J., Tang, T. & Wang, L.L. (2011). *Spectral methods: algorithms, analysis and applications* (Vol. 41). Springer Science & Business Media.

Shahmorad, S. (2005). Numerical solution of the general form linear Fredholm-Volterra integro-differential equations by the Tau method with an error estimation. *Applied Mathematics and Computation, 167(*2), 1418-1429.

Tohidi, E., Bhrawy, A.H. & Erfani, K., (2013). A collocation method based on Bernoulli operational matrix for numerical solution of generalized pantograph equation. *Appl. Math. Model.*, 37, 4283–4294.

Wei, Y. & Chen, Y. (2012). Legendre spectral collocation methods for pantograph Volterra delay-integro-differential equations. *Journal of Scientific Computing, 53*(3), 672-688.

Yuzbasi, S. (2014). Laguerre approach for solving pantograph-type Volterra integro-differential equations. *Applied Mathematics and Computation*, 232, 1183-1199.